About one multidimensional sum with Fibonacci numbers

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Abstract

This note is motivated by the following problem: Let $(f_n)_{n\geq 0}$ be the Fibonacci sequence defined by $f_0 = 0$, $f_1 = 1$ and, for all $n \geq 1$, $f_{n+1} = f_n + f_{n-1}$. Determine

$$h_n = \sum_{i,j,k} f_i f_j f_k,$$

where the sum is over $i, j, k > 0$ with $i + j + k = n$. This problem appeared in *Mathematical Horizons* [1], and is also motivated by a problem of Díaz-Barrero [2].

1 Main results

We will consider the general problem of computing the sum

$$S_m(n) := \sum_{\substack{i_1, i_2, \ldots, i_m \geq 0 \\ i_1 + i_2 + \ldots + i_m = n}} f_{i_1} f_{i_2} \ldots f_{i_m}$$

for any non negative integers m and n. (Note that $S_0(0) = 0$ as the sum is over the empty set.) It is easy to see that, in particular,

$S_m(0) = 0$ and $S_1(n) = f_n$. We have

$$
\begin{aligned}
S_m(n) &= \sum_{\substack{i_1,i_2,\ldots,i_m \geq 0 \\ i_1+i_2+\ldots+i_m=n}} f_{i_1} f_{i_2} \cdots f_{i_m} \\
&= \sum_{\substack{i_1,i_2,\ldots,i_{m-1} \geq 0 \\ i_1+i_2+\ldots+i_{m-1} \leq n}} f_{i_1} f_{i_2} \cdots f_{i_{m-1}} f_{n-\left(i_1+i_2+\ldots+i_{m-1}\right)} \\
&= \sum_{k=1}^{n-1} \sum_{\substack{i_1,i_2,\ldots,i_{m-1} \geq 0 \\ i_1+i_2+\ldots+i_{m-1}=k}} f_{i_1} f_{i_2} \cdots f_{i_{m-1}} f_{n-k}
\end{aligned}
$$

and

$$
\begin{aligned}
S_m(n+1) &= \sum_{k=1}^{n} \sum_{\substack{i_1,i_2,\ldots,i_{m-1} \geq 0 \\ i_1+i_2+\ldots+i_{m-1}=k}} f_{i_1} f_{i_2} \cdots f_{i_{m-1}} f_{n+1-k} \\
&= \sum_{k=1}^{n-1} \sum_{\substack{i_1,i_2,\ldots,i_{m-1} \geq 0 \\ i_1+i_2+\ldots+i_{m-1}=k}} f_{i_1} f_{i_2} \cdots f_{i_{m-1}} f_{n+1-k} \\
&+ \sum_{\substack{i_1,i_2,\ldots,i_{m-1} \geq 0 \\ i_1+i_2+\ldots+i_{m-1}=n}} f_{i_1} f_{i_2} \cdots f_{i_{m-1}} f_1 \\
&= \sum_{k=1}^{n-1} \sum_{\substack{i_1,i_2,\ldots,i_{m-1} \geq 0 \\ i_1+i_2+\ldots+i_{m-1}=k}} f_{i_1} f_{i_2} \cdots f_{i_{m-1}} f_{n-k} \\
&+ \sum_{k=1}^{n-1} \sum_{\substack{i_1,i_2,\ldots,i_{m-1} \geq 0 \\ i_1+i_2+\ldots+i_{m-1}=k}} f_{i_1} f_{i_2} \cdots f_{i_{m-1}} f_{n-1-k} \\
&+ \sum_{\substack{i_1,i_2,\ldots,i_{m-1} \geq 0 \\ i_1+i_2+\ldots+i_{m-1}=n}} f_{i_1} f_{i_2} \cdots f_{i_{m-1}} \\
&= \sum_{k=1}^{n-1} \sum_{\substack{i_1,i_2,\ldots,i_{m-1} \geq 0 \\ i_1+i_2+\ldots+i_{m-1}=k}} f_{i_1} f_{i_2} \cdots f_{i_{m-1}} f_{n-k}
\end{aligned}
$$

$$+\sum_{k=1}^{n-2} \sum_{\substack{i_1,i_2,\ldots,i_{m-1}\geq 0\\ i_1+i_2+\ldots+i_{m-1}=k}} f_{i_1} f_{i_2} \cdots f_{i_{m-1}} f_{n-1-k}$$

$$+ \sum_{\substack{i_1,i_2,\ldots,i_{m-1}\geq 0\\ i_1+i_2+\ldots+i_{m-1}=n}} f_{i_1} f_{i_2} \cdots f_{i_{m-1}}.$$

Thus,

$$S_m(n+1) = S_m(n) + S_m(n-1) + S_{m-1}(n), \quad n, m \in \mathbb{N}. \tag{1}$$

Using (1) we will constructively find explicit formulas for $S_2(n)$, $S_3(n)$ and $S_4(n)$. Note that, since $f_0 = 0$, then

$$h_n = S_3(n) = \sum_{\substack{i,j,k\geq 0\\ i+j+k=n}} f_i f_j f_k = \sum_{k=0}^{n} \sum_{\substack{i,j\geq 0\\ i+j=k}} f_i f_j f_{n-k}.$$

We will also use the notation g_n for $S_2(n)$ and s_n for $S_4(n)$. Namely, for $m = 1, 2, 3$, equation (1) becomes, respectively,

$$g_{n+1} = g_n + g_{n-1} + f_n, \qquad n \in \mathbb{N}, \tag{2}$$

$$h_{n+1} = h_n + h_{n-1} + g_n, \qquad n \in \mathbb{N} \tag{3}$$

and

$$s_{n+1} = s_n + s_{n-1} + h_n, \qquad n \in \mathbb{N}. \tag{4}$$

Consider now the Fibonacci operator F defined by $F(a_n) = a_{n+1} - a_n - a_{n-1}$ for any sequence $(a_n)_{n\geq 0}$ of real numbers and, in particular, for any integer k consider two special applications of operator F. Namely,

$$\begin{aligned} F(a_n f_{n+k}) &= a_{n+1} f_{n+1+k} - a_n f_{n+k} - a_{n-1} f_{n-1+k} \\ &= a_{n+1}(f_{n+1+k} - f_{n+k} - f_{n-1+k}) \\ &\quad + a_{n+1} f_{n+k} + a_{n+1} f_{n-1+k} - a_n f_{n+k} - a_{n-1} f_{n-1+k} \\ &= (a_{n+1} - a_n) f_{n+k} + (a_{n+1} - a_{n-1}) f_{n-1+k} \\ &= (a_{n+1} - a_n) f_{n+k} + (a_{n+1} - a_{n-1})(f_{n-1+k} - f_{n+k}) \\ &= (a_{n+1} - a_{n-1}) f_{n+1+k} - (a_n - a_{n-1}) f_{n+k}, \end{aligned}$$

so

$$F(a_n f_{n+k}) = (a_{n+1} - a_{n-1}) f_{n+1+k} - (a_n - a_{n-1}) f_{n+k}, \tag{5}$$

and

$$\begin{aligned} F(a_n f_{n+k+1}) &= a_{n+1} f_{n+k+2} - a_n f_{n+k+1} - a_{n-1} f_{n+k} \\ &= a_{n+1}(f_{n+k+2} - f_{n+k+1} - f_{n+k}) \\ &+ a_{n+1} f_{n+k+1} + a_{n+1} f_{n+k} - a_n f_{n+k+1} - a_{n-1} f_{n+k} \\ &= (a_{n+1} - a_n) f_{n+k+1} + (a_{n+1} - a_{n-1}) f_{n+k}, \end{aligned}$$

so

$$F(a_n f_{n+k+1}) = (a_{n+1} - a_n) f_{n+k+1} + (a_{n+1} - a_{n-1}) f_{n+k}. \tag{6}$$

Note that $F(g_n) = f_n$, $F(h_n) = g_n$ and $F(s_n) = h_n$. Note also that

$$F(a_n) = 0 \iff a_n = (a_1 - a_0) f_n + a_0 f_{n+1},$$

as can be easily proven by induction.

Now we are ready to find g_n, h_n and, afterwards, s_n. Applying (5) and (6) to $a_n = n$ we obtain, for any integer k,

$$F(n f_{n+k}) = 2 f_{n+k} - f_{n+k-1} \text{ and } F(n f_{n+k+1}) = f_{n+k+1} + 2 f_{n+k}.$$

Since for $k = 0$ we have

$$F(n f_{n+1}) = f_{n+1} + 2 f_n \quad \text{and} \quad F(n f_n) = 2 f_{n+1} - f_n,$$

thus

$$F(n f_{n+1}) + 2F(n f_n) = f_{n+1} + 2 f_n + 2(2 f_{n+1} - f_n) = 5 f_{n+1}$$

and, therefore,

$$f_{n+1} = F\left(\frac{2 f_{n+1} - n f_n}{5}\right) \tag{7}$$

using the fact F is linear.

We also have

$$2F(n f_{n+1}) - F(n f_n) = 2 f_{n+1} + 4 f_n - (2f - n + 1 - f_n) = 5 f_n,$$

from which

$$f_n = F\left(\frac{2nf_{n+1} - nf_n}{5}\right). \tag{8}$$

Hence, equation (2) is equivalent to

$$F(g_n) = F\left(\frac{2nf_{n+1} - nf_n}{5}\right) \Longleftrightarrow F\left(g_n - \frac{2nf_{n+1} - nf_n}{5}\right) = 0,$$

from which we conclude that

$$g_n = \frac{2nf_{n+1} - nf_n}{5} + c_1 f_{n+1} + c_2 f_n.$$

From the fact that $g_0 = 0$ we get $c_1 \cdot 1 + c_2 \cdot 0 = 0$ and $c_1 = 0$. Likewise, from $g_1 = 0$ we get $c_1 \cdot 1 + c_2 \cdot 1 + \frac{2\cdot 1 - 1}{5} = 0$ and $c_2 = -\frac{1}{5}$. Substituting in the above expression, we obtain

$$g_n = S_2(n) = \frac{2nf_{n+1} - (n+1)f_n}{5}, \tag{9}$$

and now we can find h_n. Indeed, applying (5) and (6) to $a_n = n^2$ we obtain

$$\begin{aligned} F(n^2 f_{n+k}) &= ((n+1)^2 - (n-1)^2)F_{n+k+1} - (n^2 - (n-1)^2)f_{n+k} \\ &= 4nf_{n+k+1} - (2n-1)f_{n+k}, \end{aligned}$$

or

$$F(n^2 f_{n+k}) = 4nf_{n+k+1} - (2n-1)f_{n+k}, \tag{10}$$

and

$$\begin{aligned} F(n^2 f_{n+k+1}) &= ((n+1)^2 - n^2)F_{n+k+1} - (n+1)^2 - (n-1)^2)f_{n+k} \\ &= (2n+1)f_{n+k+1} + 4nf_{n+k}, \end{aligned}$$

or

$$F(n^2 f_{n+k+1}) = (2n+1)f_{n+k+1} + 4nf_{n+k}. \tag{11}$$

In particular, for $k = 0$ in (10) and (11) we obtain

$$F(n^2 f_n) = 4f_{n+1} - (2n-1)f_n$$

and

$$F(n^2 f_{n+1}) = (2n+1)f_{n+1} + 4nf_n.$$

Hence,

$$\begin{aligned}F(2n^2f_n)+F(n^2f_{n+1}) &= 8nf_{n+1}-(4n-2)f_n+(2n+1)f_{n+1}+4f_n\\ &= 10nf_{n+1}+f_{n+1}+2f_n\end{aligned}$$

and

$$\begin{aligned}&10nf_{n+1}\\ &=F(2n^2f_n+n^2f_{n+1})-2f_n-f_{n+1}\\ &=F(2n^2f_n+n^2f_{n+1})\\ &-2F\left(\frac{2nf_{n+1}-nf_n}{5}\right)-F\left(\frac{nf_{n+1}+2nf_n}{5}\right)\\ &=F\left(n^2f_{n+1}+2n^2f_n-\frac{2(2nf_{n+1}-nf_n)}{5}-\frac{nf_{n+1}+2nf_n}{5}\right)\\ &=F((n^2-n)f_{n+1}+2n^2f_n),\end{aligned}$$

from which

$$nf_{n+1}=F\left(\frac{(n^2-n)f_{n+1}+2n^2f_n}{10}\right). \qquad (12)$$

Likewise, from

$$\begin{aligned}&F(2n^2f_{n+1})-F(n^2f_n)\\ &=(4n+2)f_{n+1}+8nf_n-(4nf_{n+1}-(2n-1)f_n)\\ &=10nf_n-f_n+2f_{n+1}\end{aligned}$$

we get

$$nf_n=F\left(\frac{2n^2f_{n+1}-(n^2+n)f_n}{10}\right)=F\left(\frac{ng_n}{2}\right). \qquad (13)$$

Then, using (13), (12) and (9) we obtain

$$5g_n=2nf_{n+1}-(n+1)f_n=F\left(\frac{(5n^2+3n)f_n-6nf_{n+1}}{10}\right).$$

That is, $g_n=F\left(\frac{(5n^2+3n)f_n-6nf_{n+1}}{10}\right)$ and, therefore, (2) is equivalent to $F\left(h_n-\frac{(5n^2+3n)f_n-6nf_{n+1}}{50}\right)=0$ and

$$h_n=\frac{(5n^2+3n)f_n-6nf_{n+1}}{50}+c_1f_{n+1}+c_2f_n.$$

Since $h_0 = 0 = c_1$ and $h_1 = 0 = \frac{2}{50} + c_2$, we get $c_2 = -\frac{1}{25}$, from which it follows that

$$h_n = S_3(n) = \frac{(5n^2 + 3n - 2)f_n - 6nf_{n+1}}{50}.$$

Before considering the computation of s_n we present another way to obtain g_n. Note that $F(F(g_n)) = F(f_n) = 0$ and $F(F(F(h_n))) = F(F(g_n)) = F(f_n) = 0$. Since the characteristic polynomials of $F(F(g_n))$ and $F(F(F(h_n)))$ are $(x^2 - x - 1)^2$ and $(x^2 - x - 1)^3$, respectively, then

$$g_n, h_n = P(n)\phi^n + Q(n)\overline{\phi}^n,$$

where ϕ and $\overline{\phi}$ are the roots of $x^2 - x - 1 = 0$ and P and Q are polynomials of degree at most 1 for g_n and at most 2 in the case of h_n. Since ϕ^n and $\overline{\phi}^n$ can be represented as linear combinations of f_{n+1} and f_n, then we may also represent g_n and h_n in the form $P(n)f_{n+1} + Q(n)f_n$, namely

$$g_n = (an + b)f_{n+1} + (cn + d)f_n = anf_{n+1}(cn + d)f_n$$

because $g_0 = 0$ and

$$\begin{aligned} h_n &= (an^2 + bn + c)f_{n+1} + (pn^2 + qn + r)f_n \\ &= (an^2 + bn)f_{n+1} + (pn^2 + qn + r)f_n \end{aligned}$$

because $h_0 = 0$, where $g_0 = g_1 = 0$, $g_2 = 1$, $g_3 = 2$, $h_0 = h_1 = h_2 = 0$, $h_3 = 1$ and $h_4 = 3$. Then, we get $a = \frac{2}{5}$, $c = -\frac{1}{5}$ and $d = -\frac{1}{5}$ and, therefore,

$$g_n = \frac{2nf_{n+1} - (n + 1)f_n}{5}.$$

The same expression may be obtained substituting $g_n = anf_{n+1} + (cn + d)f_n$ in $g_{n+1} - g_n - g_{n-1} = f_n$.

Now we consider the computation of $s_n = S_4(n)$. Applying (5) and (6) to $a_n = n^3$ for $k = 0$ we obtain

$$\begin{aligned} F(n^3 f_n) &= ((n + 1)^3 - (n - 1)^3)f_{n+1} - (n^3 - (n - 1)^3)f_n \\ &= (6n^2 + 2)f_{n+1} - (3n^2 - 3n + 1)f_n \end{aligned} \tag{14}$$

and

$$\begin{aligned} F(n^3 f_{n+1}) &= ((n+1)^3 - n^3) f_{n+1} - ((n+1)^3 - (n-1)^3) f_n \\ &= (3n^2 + 3n + 1) f_{n+1} + (6n^2 + 2) f_n. \end{aligned} \tag{15}$$

Since $g_n = \dfrac{2nf_{n+1} - (n+1)f_n}{5}$, $f_n = F(g_n)$, $nf_n = F\left(\dfrac{ng_n}{2}\right)$ and $nf_{n+1} = F\left(\dfrac{(n^2-n)F_{n+1} + 2n^2 f_n}{10}\right)$, then

$$\begin{aligned} & F(2n^3 f_{n+1}) - F(n^3 f_n) \\ &= (6n^2 + 6n + 2) f_{n+1} + (12n^2 + 4) f_n \\ &- ((6n^2 + 2) f_{n+1} - (3n^2 - 3n + 1) f_n) \\ &= 15n^2 f_n + 6nf_{n+1} - (3n - 5) f_n \\ &= 15n^2 f_n + 6F\left(\frac{(n^2-n)f_{n+1} + 2n^2 f_n}{10}\right) - 3F\left(\frac{ng_n}{2}\right) + 5F(g_n) \end{aligned}$$

and

$$\begin{aligned} 15n^2 f_n &= F(2n^3 f_{n+1}) - F(n^3 f_n) - 6F\left(\frac{(n^2-n)f_{n+1} + 2n^2 f_n}{10}\right) \\ &\quad + 3F\left(\frac{ng_n}{2}\right) - 5F(g_n) \\ &= F\left(\frac{(10 + 7n - 15n^2 - 10n^3) f_n + (20n^3 - 14n) f_{n+1}}{10}\right), \end{aligned}$$

from which we conclude that

$$n^2 f_n = F\left(\frac{(10 + 7n - 15n^2 - 10n^3) f_n + (20n^3 - 14n) f_{n+1}}{150}\right).$$

Since $S_3(n) = h_n = \dfrac{(5n^2 + 3n - 2) f_n - 6nf_{n+1}}{50}$,

$$nf_{n+1} = F\left(\frac{(n^2-n)f_{n+1} + 2n^2 f_n}{10}\right)$$

and

$$nf_n = F\left(\frac{2n^2 f_{n+1} - (n^2+n) f_n}{10}\right).$$

Then, from the fact that $F(S_4(n)) = S_3(n)$ we have

$$\begin{aligned} F(s_n) &= \frac{1}{10}n^2 f_n + \frac{3}{50}nf_n - \frac{1}{25}f_n - \frac{3}{25}nf_{n+1} \\ &= F\left(\frac{(11+5n-30n^2-5n^3)f_n + (10n^3-10n)f_{n+1}}{750}\right). \end{aligned}$$

Hence,

$$s_n = \frac{(11+5n-30n^2-5n^3)f_n+(10n^3-10n)f_{n+1}}{750}+c_1 f_{n+1}+c_2 f_n.$$

Since $s_0 = s_1 = 0$, then $c_1 = 0$ and $c_2 = \frac{19}{750}$ and, therefore,

$$s_n = \frac{(11+5n-30n^2-5n^3)f_n + (10n^3-10n)f_{n+1}}{750} + \frac{19}{750}f_{n+1}.$$

Finally, rearranging terms we obtain

$$s_n = S_4(n) = \frac{(n-1)(n+1)(2nf_{n+1} - (n+6)f_n)}{150}. \tag{16}$$

Remark. We claim that $S_m(n) = P_m(n)f_{n+1} + Q_m(n)f_n$, where P, Q are polynomials of degree at most m, because $F^m(S_m(n)) = 0$. Here, $F^m = F \circ F \circ \ldots \circ F$ with characteristic polynomial $(x^2 - x - 1)^m$. Then, the polynomials P and Q can be determined by substitution of $S_m(n)$ in (1) assuming that

$$S_{m-1}(n) = P_{m-1}(n)f_{n+1} + Q_{m-1}(n)f_n$$

and that we know polynomials P_{m-1} and Q_{m-1}. Using the fact that $S_m(0) = S_m(1) = 0$ we can determine both polynomials. Indeed, since

$$\begin{aligned} &P_m(n+1)f_{n+2} + Q_m(n+1)f_{n+1} - P_m(n)f_{n+1} \\ &- Q_m(n)f_n - P_m(n-1)f_n - Q_m(n-1)f_{n-1} \\ &= (P_m(n+1) - P_m(n) + Q_m(n+1) - Q_m(n-1))f_{n+1} \\ &+ (P_m(n+1) - P_m(n-1) - Q_m(n) + Q_m(n-1))f_n, \end{aligned}$$

then the fact that $F(S_m(n)) = P_{m-1}(n)f_{n+1} + Q_{m-1}(n)f_n$ implies

$$P_m(n+1) - P_m(n) + Q_m(n+1) - Q_m(n-1) = P_{m-1}(n),$$

$$P_m(n+1) - P_m(n-1) + Q_m(n) + Q_m(n-1) = Q_{m-1}(n).$$

References

[1] Bloom, D. M. “Problem 55”. *Mathematical Horizons* (1996).

[2] Díaz-Barrero, J. L. “Problem E-58”. *Arhimede math. j.* 5.1 (2018), p. 28.

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